Some applications of the generalized Bernardi - Libera - Livingston integral operator on univalent functions

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Abstract

In this paper by making use of the generalized Bernardi - Libera - Livingston integral operator we introduce and study some new subclasses of univalent functions. Also we investigate the relations between those classes and the classes which are studied by Jin-Lin Liu.

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1 Introduction

Let A be the class of functions of the form, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $U = \{z : |z| < 1\}$, also let S denote the subclass of A consisting of all univalent functions in U. Suppose λ is a real number with $0 \leq \lambda < 1$, A function $f \in S$ is said to be starlike of order λ if and only if $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \lambda, z \in U$, also $f \in S$ is said to be convex of order λ if and only if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \lambda, z \in U$, we denote by $S^*(\lambda), C(\lambda)$ the classes of starlike and convex functions of order λ respectively. It is well known that $f \in C(\lambda)$ if and only if $zf' \in S^*(\lambda)$. Let $f \in A$ and $g \in S^*(\lambda)$ then $f \in K(\beta, \lambda)$ if and only if $Re\left\{\frac{zf'(z)}{g(z)}\right\} > \beta, z \in U$ where $0 \leq \beta < 1$. These functions are called close-to-convex functions of order β type λ. A function $f \in A$ is called quasi-convex of order β type λ

if there exists a function $g \in C(\lambda)$ such that $Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \beta$. We denote this class by $K^*(\beta, \lambda)$ [10]. It is easy to see that $f \in K^*(\beta, \gamma)$ if and only if $zf' \in K(\beta, \gamma)$ [9]. For $f \in A$ if for some $\lambda (0 \leq \lambda < 1)$ and $\eta (0 < \eta \leq 1)$ we have

$$\left| arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| < \frac{\pi}{2}\eta, \quad z \in U \tag{1.1}$$

then $f(z)$ is said to be strongly starlike of order η and type λ in U and we denote this class by $S^*(\eta, \lambda)$. If $f \in A$ satisfies the condition

$$\left| arg\left(1 + \frac{zf''(z)}{f'(z)} - \lambda\right)\right| < \frac{\pi}{2}\eta, \quad z \in U \tag{1.2}$$

for some λ and η as above then we say that $f(z)$ is strongly convex of order η and type λ in U and we denote this class by $C(\eta, \lambda)$. Clearly $f \in C(\eta, \lambda)$ if and only if $zf' \in S^*(\eta, \lambda)$, specially we have $S^*(1, \lambda) = S^*(\lambda)$ and $C(1, \lambda) = C(\lambda)$.

For $c > -1$ and $f \in A$ the generalized Bernardi - Libera - Livingston integral operator $L_c f$ is defined as follows

$$L_c f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt. \tag{1.3}$$

This operator for $c \in N = \{1, 2, 3, \cdots\}$ was studied by Bernardi [1] and for $c = 1$ by Libera [5] (see also [8]). Now by making use of the operator given by (1.3) we introduce the following classes.

$$\begin{aligned}
S_c^*(\lambda) &= \{f \in A : L_c f \in S^*(\lambda)\} \\
C_c(\lambda) &= \{f \in A : L_c f \in C(\lambda)\} \\
K_c(\beta, \lambda) &= \{f \in A : L_c f \in K(\beta, \lambda)\} \\
K_c^*(\beta, \lambda) &= \{f \in A : L_c f \in K^*(\beta, \lambda)\} \\
ST_c(\eta, \lambda) &= \left\{f \in A : L_c f \in S^*(\eta, \lambda), \frac{z(L_c f(z))'}{L_c f(z)} \neq \lambda, z \in U\right\} \\
CV_c(\eta, \lambda) &= \left\{f \in A : L_c f \in C(\eta, \lambda), \frac{(z(L_c f(z))')'}{(L_c f(z))'} \neq \lambda, z \in U\right\}.
\end{aligned}$$

Obviously $f \in CV_c(\eta,\lambda)$ if and only if $zf' \in ST_c(\eta,\lambda)$. J. L. Liu [6] and [7] introduced and investigated similarly the classes $S^*_\sigma(\lambda), C_\sigma(\lambda), K_\sigma(\beta,\lambda), K^*_\sigma(\beta,\lambda), ST_\sigma(\eta,\lambda), CV_\sigma(\eta,\lambda)$ by making use of the integral operator $I^\sigma f$ given by

$$I^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)} \int\limits_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} f(t)dt, \sigma > 0, f \in A. \tag{1.4}$$

The operator I^σ is introduced by Jung, Kim and Srivastava [3] and then investigated by Uralogaddi and Somanatha [12], Li [4] and Liu [6]. For the integral operators given by (1.3) and (1.4) we have easily verified following relationships.

$$I^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n \tag{1.5}$$

$$L_c f(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{n+c} a_n z^n \tag{1.6}$$

$$z(I^\sigma L_c f(z))' = (c+1) I^\sigma f(z) - c I^\sigma L_c f(z) \tag{1.7}$$

$$z(L_c I^\sigma f(z))' = (c+1) I^\sigma f(z) - c L_c I^\sigma f(z). \tag{1.8}$$

It follows from (1.5) that one can define the operator I^σ for any real number σ. In this paper we investigate the properties of the classes $S^*_c(\lambda), C_c(\lambda), K_c(\beta,\lambda), K^*_c(\beta,\lambda)$, $ST_c(\eta,\lambda), CV_c(\eta,\lambda)$, also we study the relations between these classes by the classes which are introduced by Liu in [6] and [7]. For our purposes we need the following lemmas.

Lemma 1.1 [9]. Let $u = u_1 + iu_2, v = v_1 + iv_2$ and let $\psi(u,v)$ be a complex function $\psi : D \subset \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. Suppose that ψ satisfies the following conditions

(i) $\psi(u,v)$ is continuous in D

(ii) $(1,0) \in D$ and $Re\{\psi(1,0)\} > 0$

(iii) $Re\{\psi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$.

Let $p(z) = 1 + \sum\limits_{n=2}^{\infty} c_n z^n$ be analytic in U so that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $Re\{\psi(p(z), zp'(z))\} > 0, z \in U$ then $Re\{p(z)\} > 0, z \in U$.

Lemma 1.2 [11]. Let the function $p(z) = 1+\sum_{n=1}^{\infty} c_n z^n$ be analytic in U and $p(z) \neq 0, z \in U$ if there exists a point $z_0 \in U$ such that $|arg(p(z))| < \frac{\pi}{2}\eta$ for $|z| < |z_0|$ and $|arg\ p(z_0)| = \frac{\pi}{2}\eta$ where $0 < \eta \leq 1$ then $\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta$ and $k \geq \frac{1}{2}(r + \frac{1}{r})$ when $arg\ p(z_0) = \frac{\pi}{2}\eta$ also $k \leq \frac{-1}{2}(r + \frac{1}{r})$ when $arg\ p(z_0) = \frac{-\pi}{2}\eta$, and $p(z_0)^{1/\eta} = \pm ir (r > 0)$.

2 Main Results

In this section we obtain some inclusion theorems.

Theorem 2.1 : (i) For $f \in A$ if $Re\left\{\frac{zf'(z)}{f(z)} - \frac{z(L_c f(z))'}{L_c f(z)}\right\} > 0$, then $S_c^*(\lambda) \subset S_{c+1}^*(\lambda)$.

(ii) For $f \in A$ if $Re\left\{\frac{zf'(z)}{f(z)} - \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}\right\} > 0$ then $S_{c+1}^*(\lambda) \subset S_c^*(\lambda)$.

Proof : (i) Suppose that $f \in S_c^*(\lambda)$ and set

$$\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} - \lambda = (1-\lambda)p(z) \tag{2.1}$$

where $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$. An easy calculation shows that

$$\frac{\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}\left[2 + c + \frac{z(L_{c+1}f(z))''}{(L_{c+1}f(z))'}\right]}{\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} + c + 1} = \frac{zf'(z)}{f(z)}. \tag{2.2}$$

By setting $H(z) = \frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}$ we have

$$1 + \frac{z(L_{c+1}f(z))''}{(L_{c+1}f(z))'} = H(z) + \frac{zH'(z)}{H(z)}. \tag{2.3}$$

By making use of (2.3) in (2.2) since $H(z) = \lambda + (1-\lambda)p(z)$ so we obtain

$$(1-\lambda)p(z) + \frac{(1-\lambda)zp'(z)}{\lambda + c + 1 + (1-\lambda)p(z)} = \frac{zf'(z)}{f(z)} - \lambda. \tag{2.4}$$

If we consider $\psi(u,v) = (1-\lambda)u + \frac{(1-\lambda)v}{\lambda+c+1+(1-\lambda)u}$ then $\psi(u,v)$ is a continuous function in $D = \left\{\mathbb{C} - \frac{\lambda+c+1}{\lambda-1}\right\} \times \mathbb{C}$ and $(1,0) \in D$ also $\psi(1,0) > 0$ and for all $(iu_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$ we have

$$Re\ \ \psi(iu_2, v_1) = \frac{(1-\lambda)(\lambda+c+1)v_1}{(1-\lambda)^2 u_2^2 + (\lambda+c+1)^2} \leq \frac{-(1-\lambda)(\lambda+c+1)(1+u_2^2)}{2[(1-\lambda)^2 u_2^2 + (\lambda+c+1)^2]} < 0.$$

Therefore the function $\psi(u,v)$ satisfies the conditions of Lemma 1.1 and since in view of the assumption by considering (2.4) we have $Re\{\psi(p(z),zp'(z))\} > 0$ therefore Lemma 1 implies that $Re\ p(z) > 0, z \in U$ and this completes the proof.

(ii) For proving this part of theorem by the same method and using the easily verified formula similar to (2.2) by replacing $c+1$ with c we get the desired result.

Theorem 2.2 : (i) For $f \in A$ if $Re\left\{\frac{zf'(z)}{f(z)} - \frac{z(L_c f(z))'}{L_c f(z)}\right\} > 0$ then $C_c(\lambda) \subset C_{c+1}(\lambda)$.

(ii) For $f \in A$ if $Re\left\{\frac{zf'(z)}{f(z)} - \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}\right\} > 0$ then $C_{c+1}(\lambda) \subset C_c(\lambda)$.

Proof : (i) In view of part (i) of Theorem 1 we can write

$f \in C_c(\lambda) \Leftrightarrow L_c f \in C(\lambda) \Leftrightarrow z(L_c f)' \in S^*(\lambda) \Leftrightarrow L_c zf' \in S^*(\lambda) \Leftrightarrow zf' \in S_c^*(\lambda) \Rightarrow zf' \in S_{c+1}^*(\lambda) \Leftrightarrow L_{c+1} zf' \in S^*(\lambda) \Leftrightarrow z(L_{c+1} f)' \in S^*(\lambda) \Leftrightarrow L_{c+1} f \in C(\lambda) \Leftrightarrow f \in C_{c+1}(\lambda)$.

By the similar way we can prove the part (ii) of theorem.

Theorem 2.3 : If $c \geq -\lambda$ then $f \in S^*(\lambda)$ implies $f \in S_c^*(\lambda)$.

Proof : By differentiating logarithmically from both sides of (1.3) with respect to z we obtain

$$\frac{z(L_c f(z))'}{L_c f(z)} + c = \frac{(c+1)f(z)}{L_c f(z)}. \tag{2.5}$$

Again differentiating logarithmically from both sides of (2.5) we have

$$p(z) + \frac{zp'(z)}{c+\lambda+p(z)} = \frac{zf'(z)}{f(z)} - \lambda \tag{2.6}$$

where $p(z) = \frac{z(L_c f(z))'}{L_c f(z)} - \lambda$. Let us consider $\psi(u,v) = u + \frac{v}{u+c+\lambda}$, then ψ is a continuous function in $D = \{\mathbb{C} - (-c-\lambda)\} \times \mathbb{C}$ and $(1,0) \in D$ also $Re\ \psi(1,0) > 0$. If $(iu_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$ then $Re\ \psi(iu_2, v_1) = \frac{v_1(c+\lambda)}{u_2^2+(c+\lambda)^2} \leq 0$, also since $f \in S^*(\lambda)$ then (2.6) gives $Re(\psi(p(z), zp'(z))) = Re\left\{\frac{zf'(z)}{f(z)} - \lambda\right\} > 0$. Therefore Lemma 1 concludes that $Re\{p(z)\} > 0$ and this completes the proof.

Corollary 2.4 : If $c \geq \lambda$ then $f \in C(\lambda)$ implies $f \in C_c(\lambda)$.

Proof : We have

$f \in C(\lambda) \Leftrightarrow zf' \in S^*(\lambda) \Longrightarrow zf' \in S_c^*(\lambda) \Leftrightarrow L_c zf' \in S^*(\lambda) \Leftrightarrow z(L_c f)' \in S^*(\lambda) \Leftrightarrow L_c f \in C(\lambda) \Leftrightarrow f \in C_c(\lambda)$.

Theorem 2.5 : (i) For $f \in A$ if $\left|arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| \leq \left|arg\left(\frac{z(L_c f(z))'}{L_c f(z)} - \lambda\right)\right|, z \in U$ then $ST_c(\eta, \lambda) \subset ST_{c+1}(\eta, \lambda), c > -1$.

(ii) For $f \in A$ if $\left|arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| \leq \left|arg\left(\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} - \lambda\right)\right|, z \in U$ then $ST_{c+1}(\eta, \lambda) \subset ST_c(\eta, \lambda), c > -1$.

Proof : (i) Let $f \in ST_c(\eta, \lambda)$ and put

$$\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} = \lambda + (1-\lambda)p(z) \tag{2.7}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is analytic in U with $p(z) \neq 0, z \in U$. It is easy to see that

$$z(L_{c+1}f(z))' + (c+1)L_{c+1}f(z) = (c+2)f(z). \tag{2.8}$$

Differentiating logarithmically with respect to z from both sides of (2.8) gives

$$\frac{z\left(\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}\right)'}{\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} + c + 1} + \frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} = \frac{zf'(z)}{f(z)}. \tag{2.9}$$

Now by making use of (2.7) in (2.9) we have

$$\frac{(1-\lambda)zp'(z)}{\lambda + c + 1 + (1-\lambda)p(z)} + (1-\lambda)p(z) = \frac{zf'(z)}{f(z)} - \lambda. \tag{2.10}$$

Suppose that there exists $z_0 \in U$ in such a way $|arg(p(z))| < \frac{\pi}{2}\eta$ for $|z| < |z_0|$ and $|arg(p(z_0))| = \frac{\pi}{2}\eta$, then by Lemma 1.2 we have $\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta$ and $p(z_0)^{1/\eta} = \pm ir (r > 0)$ where $k \geq \frac{1}{2}(r + \frac{1}{r})$ when $arg(p(z_0)) = \frac{\pi}{2}\eta$ and $k \leq \frac{-1}{2}(r + \frac{1}{r})$ when $arg(p(z_0)) = \frac{-\pi}{2}\eta$. If

$p(z_0)^{1/\eta} = ir$ then $arg(p(z_0)) = \frac{\pi}{2}\eta$ and by considering (2.10) we have

$$\left|arg\left(\frac{z_0(L_cf(z_0))'}{L_cf(z_0)} - \lambda\right)\right| \geq arg\left(\frac{z_0f'(z_0)}{f(z_0)} - \lambda\right)$$
$$= arg\left\{(1-\lambda)p(z_0)\left[1 + \frac{ik\eta}{\lambda+c+1+(1-\lambda)r^\eta e^{i\frac{\pi}{2}\eta}}\right]\right\}$$
$$= \frac{\pi}{2}\eta$$
$$+\tan^{-1}\left\{\frac{k\eta[\lambda+c+1+r^\eta(1-\lambda)\cos\frac{\pi}{2}\eta]}{(\lambda+c+1)^2+r^{2\eta}(1-\lambda)^2+(1-\lambda)(\lambda+c+1)\cos\frac{\pi}{2}\eta+k\eta r^\eta(1-\lambda)\sin\frac{\pi}{2}\eta}\right\}$$
$$\geq \frac{\pi}{2}\eta \quad (\text{Because} \quad k \geq \frac{1}{2}(r+\frac{1}{r}) \geq 1)$$

which is a contradiction by $f(z) \in ST_c(\eta,\lambda)$.

Now suppose $p(z_0)^{1/\eta} = -ir$ then $arg(p(z_0)) = \frac{-\pi}{2}\eta$ and we have

$$-\left|arg\left(\frac{z_0(L_cf(z_0))'}{L_cf(z_0)} - \lambda\right)\right| \leq arg\left(\frac{z_0f'(z_0)}{f(z_0)} - \lambda\right)$$
$$= \frac{-\pi}{2}\eta + arg\left\{1 + \frac{ik\eta}{\lambda+c+1+(1-\lambda)r^\eta e^{-i\frac{\pi}{2}\eta}}\right\}$$
$$= \frac{-\pi}{2}\eta$$
$$+\tan^{-1}\left\{\frac{k\eta[\lambda+c+1+r^\eta(1-\lambda)\cos\frac{\pi}{2}\eta]}{(\lambda+c+1)^2+r^{2\eta}(1-\lambda)^2+2r^\eta(1-\lambda)(\lambda+c+1)\cos\frac{\pi}{2}\eta-k\eta r^\eta(1-\lambda)\sin\frac{\pi}{2}\eta}\right\}$$
$$\leq \frac{-\pi}{2}\eta \quad (\text{Because} \quad k \leq \frac{-1}{2}(r+\frac{1}{r}) \leq -1)$$

which contradicts our assumption that $f \in ST_c(\eta,\lambda)$, therefore $|arg(p(z))| < \frac{\pi}{2}, z \in U$ and finally $\left|arg\left(\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} - \lambda\right)\right| < \frac{\pi}{2}\eta, z \in U$. However since for every $\lambda (0 \leq \lambda < 1)$ we have $\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} \neq \lambda$ thus we have $f \in ST_{c+1}(\eta,\lambda)$ and the proof is complete.

(ii) The proof of this part of theorem is similar with the proof of part (i) and we omit the proof.

Corollary 2.6 : (i) For $f \in A$ if $\left|arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| \leq \left|arg\left(\frac{z(L_cf(z))'}{L_cf(z)} - \lambda\right)\right|, z \in U$ then $CV_c(\eta,\lambda) \subset CV_{c+1}(\eta,\lambda)$.

(ii) For $f \in A$ if, $\left|arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| \leq \left|arg\left(\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} - \lambda\right)\right|, z \in U$ then we have $CV_{c+1}(\eta,\lambda) \subset CV_c(\eta,\lambda)$.

Proof : We give only the proof of part (i) and for this we have

$f \in CV_c(\eta,\lambda) \Leftrightarrow L_cf \in C(\eta,\lambda) \Leftrightarrow z(L_cf)' \in S^*(\eta,\lambda) \Leftrightarrow L_czf' \in S^*(\eta,\lambda) \Leftrightarrow zf' \in ST_c(\eta,\lambda) \Longrightarrow zf' \in ST_{c+1}(\eta,\lambda) \Leftrightarrow L_{c+1}zf' \in S^*(\eta,\lambda) \Leftrightarrow z(L_{c+1}f)' \in S^*(\eta,\lambda) \Leftrightarrow L_{c+1}f \in C(\eta,\lambda) \Leftrightarrow f \in CV_{c+1}(\eta,\lambda)$.

Theorem 2.7 : For every $c > -1$ we have $CV_c(\eta,\lambda) \subset ST_c(\eta,\lambda)$.

Proof : Let $f \in CV_c(\eta,\lambda)$ then $\left|arg\left(1+\frac{z(L_cf(z))''}{(L_cf(z))'}-\lambda\right)\right| < \frac{\pi}{2}\eta, z \in U$ and $\frac{(z(L_cf(z))')'}{(L_cf(z))'} \neq \lambda, z \in U$. Suppose that

$$\frac{z(L_cf(z))'}{L_cf(z)} = \lambda + (1-\lambda)p(z) \tag{2.11}$$

where $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ is analytic in U with $p(z) \neq 0$ for all $z \in U$. Differentiating both sides of (2.11) logarithmically with respect to z gives

$$1 + \frac{z(L_cf(z))''}{(L_cf(z))'} - \lambda = (1-\lambda)p(z) + \frac{(1-\lambda)zp'(z)}{\lambda+(1-\lambda)p(z)}.$$

If there exists a point $z_0 \in U$ such that $|arg(p(z))| < \frac{\pi}{2}\eta(|z| < |z_0|)$ and $|arg(p(z_0))| = \frac{\pi}{2}\eta$ then by Lemma 2 we obtain $\frac{z_0p'(z_0)}{p(z_0)} = ik\eta$ and $p(z_0)^{1/\eta} = \pm ir(r > 0)$ where $k \geq \frac{1}{2}(r+\frac{1}{r})$ when $arg(p(z_0)) = \frac{\pi}{2}\eta$ and $k \leq -\frac{1}{2}(r+\frac{1}{r})$ when $arg(p(z_0)) = -\frac{\pi}{2}\eta$. Suppose that $arg(p(z_0)) = \frac{-\pi}{2}\eta$ then

$$\begin{aligned}
&arg\left\{1+\frac{z_0(L_cf(z_0))''}{(L_cf(z_0))'}-\lambda\right\}\\
&= arg\left\{(1-\lambda)r^\eta e^{-i\frac{\pi}{2}\eta}\left[1+\frac{ik\eta}{\lambda+(1-\lambda)r^\eta e^{-i\frac{\pi}{2}\eta}}\right]\right\}\\
&= \frac{-\pi}{2}\eta + arg\left\{1+\frac{ik\eta}{\lambda+(1-\lambda)r^\eta e^{-i\frac{\pi}{2}\eta}}\right\}\\
&= \frac{-\pi}{2}\eta + \tan^{-1}\left\{\frac{k\eta[\lambda+(1-\lambda)r^\eta\cos\frac{\pi}{2}\eta]}{\lambda^2+2\lambda(1-\lambda)r^\eta\cos\frac{\pi}{2}\eta+(1-\lambda)^2r^{2\eta}-k\eta(1-\lambda)r^\eta\sin\frac{\pi}{2}\eta}\right\}\\
&\leq \frac{-\pi}{2}\eta \quad (\text{Because } \; k \leq \frac{-1}{2}(r+\frac{1}{r}) \leq -1)
\end{aligned}$$

which is a contradiction by $f \in CV_c(\eta,\lambda)$. For the case $arg(p(z_0)) = \frac{\pi}{2}\eta$ by the same way

and considering $k \geq \frac{1}{2}(r + \frac{1}{r}) \geq 1$ we obtain

$$arg\left\{1 + \frac{z_0(L_c f(z_0))''}{(L_c f(z_0))'} - \lambda\right\} \geq -\frac{\pi}{2}\eta.$$

This also contradicts our assumption that $f \in CV_c(\eta, \lambda)$, thus we have $|arg(p(z))| < \frac{\pi}{2}\eta (z \in U)$ and finally

$$\left|arg\left(\frac{z(L_c f(z))'}{L_c f(z)} - \lambda\right)\right| < \frac{\pi}{2}\eta, \quad z \in U.$$

Theorem 2.8 : (i) If for every $f \in A$ and $g \in S_c^*(\lambda)$ we have

$$Re\left\{\frac{z\frac{d}{dz}\left(\frac{L_c zf'(z)}{L_c g(z)}\right)}{\frac{z(L_c g(z))'}{L_c g(z)} + c}\right\} > 0 \tag{2.12}$$

and

$$Re\left\{\frac{zg'(z)}{g(z)} - \frac{z(L_c g(z))'}{L_c g(z)}\right\} > 0 \tag{2.13}$$

then $K_c(\beta, \lambda) \subset K_{c+1}(\beta, \lambda)$.

(ii) If for every $f \in A$ and $g \in S^*(\lambda)$ we have

$$Re\left\{\frac{z\frac{d}{dz}\left(\frac{L_{c+1} zf'(z)}{L_{c+1} g(z)}\right)}{\frac{z(L_{c+1} g(z))'}{L_{c+1} g(z)} + c}\right\} > 0 \tag{2.14}$$

and

$$Re\left\{\frac{zg'(z)}{g(z)} - \frac{z(L_{c+1} g(z))'}{L_{c+1} g(z)}\right\} > 0 \tag{2.15}$$

then $K_{c+1}(\beta, \lambda) \subset K_c(\beta, \lambda)$.

Proof : (i) Let $f \in K_c(\beta, \lambda)$ then there exists a function $\varphi(z) \in S^*(\lambda)$ such that

$$Re\left\{\frac{z(L_c f(z))'}{\varphi(z)}\right\} > \beta, \quad z \in U.$$

There is a function g in such a way $L_c g(z) = \varphi(z)$ therefore $g \in S_c^*(\lambda)$ and we have $Re\left\{\frac{z(L_c f(z))'}{L_c g(z)}\right\} > \beta, z \in U$. Suppose that

$$\frac{z(L_{c+1} f(z))'}{L_{c+1} g(z)} - \beta = (1 - \beta)p(z) \tag{2.16}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Now in view of (2.12) we can write

$$
\begin{aligned}
0 > Re\left\{\frac{-z\frac{d}{dz}\left(\frac{L_c z f'(z)}{L_c g(z)}\right)}{\frac{z(L_c g(z))'}{L_c g(z)} + c}\right\} &= Re\left\{\frac{zL_c zf'(z)(L_c g(z))' - z(L_c zf'(z)' L_c g(z)}{L_c g(z)[z(L_c g(z))' + cL_c g(z)}\right\} \\
= Re\left\{\frac{z(L_c f(z))'[z(L_c g(z))' + cL_c g(z)] - L_c g(z)[z(L_c zf'(z))' + cL_c zf'(z)]}{L_c g(z)[z(L_c g(z))' + cL_c g(z)]}\right\} \\
= Re\left\{\frac{z(L_c f(z))'}{L_c g(z)}\right\} - Re\left\{\frac{cL_c zf'(z) + z(L_c zf'(z))'}{z(L_c g(z))' + cL_c g(z))}\right\}.
\end{aligned}
$$

Therefore we have

$$
Re\left\{\frac{z(L_c f(z))'}{L_c g(z)}\right\} < Re\left\{\frac{z(L_c zf'(z))' + c(L_c zf'(z))}{z(L_c g(z))' + cL_c g(z)}\right\} \tag{2.17}
$$

Now by easy computation we obtain the following identities.

$$
z(L_c zf'(z))' + c(L_c zf'(z)) = \frac{c+1}{c+2}[z(L_{c+1} zf'(z))' + (c+1)(L_{c+1} zf'(z))] \tag{2.18}
$$

$$
z(L_c g(z))' + c(L_c g(z) = \frac{c+1}{c+2}[z(L_{c+1} g(z))' + (c+1)(L_{c+1} g(z))]. \tag{2.19}
$$

By making use of (2.18) and (2.19) in (2.17) we get

$$
\begin{aligned}
Re\left\{\frac{z(L_c f(z))'}{L_c g(z)}\right\} &< Re\frac{z(L_{c+1} zf'(z))' + (c+1)(L_{c+1} zf'(z))}{z(L_{c+1} g(z))' + (c+1)L_{c+1} g(z)} \\
= Re\frac{\frac{z(L_{c+1} zf'(z))'}{L_{c+1} g(z)} + (c+1)\frac{z(L_{c+1} f(z))'}{L_{c+1} g(z)}}{\frac{z(L_{c+1} g(z))'}{L_{c+1} g(z)} + c + 1}.
\end{aligned}
$$

In view of (2.13) and considering Theorem 1 we have $g \in S^*_{c+1}(\lambda)$ and $\frac{z(L_{c+1} g(z))'}{L_{c+1} g(z)} = (1-\lambda)Q(z) + \lambda$ where $Re(Q(z)) > 0, z \in U$, also according to (2.16) we have

$$
L_{c+1} zf'(z) = L_{c+1} g(z)[(1-\beta)p(z) + \beta]. \tag{2.20}
$$

Differentiating logarithmically with respect to z from both sides of (2.20) gives

$$
\frac{z(L_{c+1} zf'(z))'}{L_{c+1} g(z)} = (1-\beta)zp'(z) + [(1-\lambda)Q(z) + \lambda][(1-\beta)p(z) + \beta]. \tag{2.21}
$$

However,

$$
\begin{aligned}
&Re\left\{\frac{z(L_cf(z))'}{L_cg(z)}\right\}\\
&< Re\frac{(1-\beta)zp'(z)+[(1-\lambda)Q(z)+\lambda][(1-\beta)p(z)+\beta]+(c+1)[(1-\beta)p(z)+\beta]}{(1-\lambda)Q(z)+\lambda+c+1}\\
&= Re\{(1-\beta)p(z)+\beta\}+Re\frac{(1-\beta)zp'(z)}{(1-\lambda)Q(z)+\lambda+c+1}.
\end{aligned}
$$

Equivalently

$$
Re\left\{\frac{z(L_cf(z))'}{L_cg(z)}-\beta\right\}< Re\left\{(1-\beta)p(z)+\frac{(1-\beta)zp'(z)}{(1-\lambda)Q(z)+\lambda+c+1}\right\}. \tag{2.22}
$$

By considering the function $\psi(u,v)$ as

$$
\psi(u,v)=(1-\beta)u+\frac{(1-\beta)v}{(1-\lambda)Q(z)+\lambda+c+1}
$$

and noting that $Re(Q(z))>0$ we can easily verify that the function ψ is a continuous function in $D=\mathbb{C}\times\mathbb{C}$ and $Re\{\psi(1,0)\}>0$, also if $v_1\le -\frac{1}{2}(1+u_2^2)$ then we have

$$
\begin{aligned}
&Re\{\psi(iu_2,v_1)= Re\left\{(1-\beta)iu_2+\frac{(1-\beta)v_1}{(1-\lambda)Q(z)+\lambda+c+1}\right\}\\
&= Re\left\{\frac{(1-\beta)v_1[\lambda+c+1+(1-\lambda)Re(Q(z))-i(1-\lambda)I_m(Q(z))]}{[\lambda+c+1+(1-\lambda)Re(Q(z))]^2+[(1-\lambda)I_m(Q(z))]^2}\right\}\\
&= \frac{(1-\beta)v_1[\lambda+c+1+(1-\lambda)Re(Q(z))]}{[\lambda+c+1+(1-\lambda)Re(Q(z))]^2+[(1-\lambda)I_m(Q(z)]^2}\\
&\le \frac{-(1-\beta)(1+u_2^2)[\lambda+c+1+(1-\lambda)Re(Q(z))]}{[\lambda+c+1+(1-\lambda)Re(Q(z))]^2+[(1-\lambda)I_m(Q(z))]^2}<0.
\end{aligned}
$$

Finally since in view of (2.22) we have $Re\{\psi(p(z),zp'(z))\}>0$ therefore Lemma 1.1 gives $Re(p(z))>0, z\in U$ and the proof is complete.

The proof of part (ii) is similar to part (i) and we omit it.

By the same method used in Theorem 6 we can prove the next theorem.

Theorem 2.9 : (i) If for every $f\in A$ and $g\in C_c(\lambda)$ we have

$$
Re\left\{\frac{z\frac{d}{dz}\left(\frac{(L_czf'(z))'}{(L_cg(z))'}\right)}{\frac{z(L_cg(z))''}{(L_cg(z))'}+c+1}\right\}>0
$$

and

$$Re\left\{\frac{zg'(z)}{g(z)}-\frac{z(L_cg(z))'}{L_cg(z)}\right\}>0$$

then $K^*_c(\beta,\lambda)\subset K^*_{c+1}(\beta,\lambda)$.

(ii) If for every $f\in A$ and $g\in C_{c+1}(\lambda)$ we have

$$Re\left\{\frac{z\frac{d}{dz}\left(\frac{(L_{c+1}zf'(z))'}{(L_{c+1}g(z))'}\right)}{\frac{z(L_{c+1}g(z))''}{(L_{c+1}g(z))'}+c+1}\right\}>0$$

and

$$Re\left\{\frac{zg'(z)}{g(z)}-\frac{z(L_{c+1}g(z))'}{L_{c+1}g(z)}\right\}>0$$

then $K^*_{c+1}(\beta,\lambda)\subset K^*_c(\beta,\lambda)$.

Theorem 2.10 : If $-\lambda\le c\le 1-2\lambda$ then $f\in S^*_\sigma(\lambda)$ implies $I^\sigma f\in S^*_c(\lambda)$.

Proof : Suppose that $f\in S^*_\sigma(\lambda)$ and set

$$\frac{z(L_cI^\sigma f(z))'}{L_cI^\sigma f(z)}=\frac{1+(1-2\lambda)w(z)}{1-w(z)},\quad z\in U \tag{2.23}$$

where $w(z)$ is analytic or meromorphic in U with $w(0)=0$. By using (1.8) and (2.23) we obtain

$$\frac{I^\sigma f(z)}{L_cI^\sigma f(z)}=\frac{c+1+(1-c-2\lambda)w(z)}{(c+1)(1-w(z))}. \tag{2.24}$$

Differentiating logarithmically both sides of (2.24) with respect to z gives

$$\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}=\frac{1+(1-2\lambda)w(z)+zw'(z)}{1-w(z)}+\frac{(1-c-2\lambda)zw'(z)}{c+1+(1-c-2\lambda)w(z)}$$

Now we assert that $|w(z)|<1, z\in U$, if not then there exists a point $z_0\in U$ such that $\max\limits_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1$ therefore by Jacks' Lemma we have $z_0w'(z_0)=kw(z_0), k\ge 1$. So we have

$$\begin{aligned}&Re\left\{\frac{z_0(I^\sigma f(z_0))'}{I^\sigma f(z_0)}-\lambda\right\}\\&=Re\left\{\frac{1+(1-2\lambda+k)e^{i\theta}}{1-e^{i\theta}}+\frac{(1-c-2\lambda)ke^{i\theta}}{c+1+(1-c-2\lambda)e^{i\theta}}-\lambda\right\}\\&=\frac{-2k(1-\lambda)(c+\lambda)}{(1+c)^2+2(1+c)(1-c-2\lambda)\cos\theta+(1-c-2\lambda)^2}\le\frac{-k(c+\lambda)}{2(1-\lambda)}\le 0.\end{aligned}$$

This contradicts our hypothesis $f \in S^*_\sigma(\lambda)$ thus $|w(z)| < 1, z \in U$ and by cosidering (2.23) we conclude that $I^\sigma f \in S^*_c(\lambda)$.

Corollary 2.11 : If $-\lambda < c < 1 - 2\lambda$ and $f \in C_\sigma(\lambda)$ then $I^\sigma f \in C_c(\lambda)$.

Proof : We have

$f \in C_\sigma(\lambda) \Leftrightarrow zf' \in S^*_\sigma(\lambda) \Longrightarrow I^\sigma(zf') \in S^*_c(\lambda) \Leftrightarrow z(I^\sigma f)' \in S^*_c(\lambda) \Leftrightarrow I^\sigma f \in C_c(\lambda)$.

Theorem 2.12 : Let $-\lambda \le c, 0 \le \lambda < 1$. If $f \in A$ and $\frac{z(L_c I^\sigma f(z))'}{L_c I^\sigma f(z)} \neq \lambda, z \in U$ then $f \in ST_\sigma(\eta, \lambda)$ implies that $I^\sigma f \in ST_c(\eta, \lambda)$.

Proof : Let $f \in ST_\sigma(\eta, \lambda)$ and put

$$\frac{z(L_c I^\sigma f(z))'}{L_c I^\sigma f(z)} = \lambda + (1-\lambda)p(z) \tag{2.25}$$

where $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ and $p(z) \neq 0, z \in U$. By considering (1.8) and (2.25) we have

$$(c+1)\frac{I^\sigma f(z)}{L_c I^\sigma f(z)} = c + \lambda + (1-\lambda)p(z) \tag{2.26}$$

Differentiating logarithmically with respect to z from both sides of (2.26) gives

$$\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} - \lambda = (1-\lambda)p(z) + \frac{(1-\lambda)zp'(z)}{c + \lambda + (1-\lambda)p(z)}.$$

Suppose that there eixsts a point $z_0 \in U$ such that $|arg(p(z))| < \frac{\pi}{2}\eta(|z| < |z_0|)$ and $|arg(p(z_0))| = \frac{\pi}{2}\eta$ then by Lemma 1.2 we have $\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta$ and $p(z_0)^{1/\eta} = \pm\ ir(r > 0)$. If $p(z_0)^{1/\eta} = ir$ then

$$\begin{aligned}
&\frac{z_0(I^\sigma f(z_0))'}{I^\sigma f(z_0)} - \lambda = (1-\lambda)p(z_0)\left[1 + \frac{\frac{z_0 p'(z_0)}{p(z_0)}}{c + \lambda + (1-\lambda)p(z_0)}\right] \\
&= (1-\lambda)r^\eta e^{i\frac{\pi}{2}\eta}\left[1 + \frac{ik\eta}{c + \lambda + (1-\lambda)r^\eta e^{i\frac{\pi}{2}\eta}}\right] \\
&= \frac{\pi}{2}\eta + arg\left\{1 + \frac{ik\eta}{c + \lambda + (1-\lambda)r^\eta e^{i\frac{\pi}{2}\eta}}\right\} \\
&= \frac{\pi}{2}\eta \\
&+ \tan^{-1}\left\{\frac{k\eta[c + \lambda + (1-\lambda)r^\eta \cos\frac{\pi}{2}\eta]}{(c+\lambda)^2 + 2(c+\lambda)(1-\lambda)r^\eta\cos\frac{\pi}{2}\eta + (1-\lambda)^2 r^{2\eta} + k\eta(1-\lambda)r^\eta \sin\frac{\pi}{2}\eta}\right\} \\
&\ge \frac{\pi}{2}\eta \quad (\text{Because} \quad k \ge \frac{1}{2}(r + \frac{1}{r}) \ge 1)
\end{aligned}$$

which contradicts our assumption $f \in ST_\sigma(\eta, \lambda)$. By the same method we get a contradiction for the case $p(z_0)^{1/\eta} = -ir(r > 0)$, therefore we have $|arg(p(z))| < \frac{\pi}{2}\eta, z \in U$ and in view of (2.14) we conclude that $I^\sigma f \in ST_c(\eta, \lambda)$.

Corollary 2.13 : Let $c \geq \lambda, 0 \leq \lambda < 1$. If $f \in A$ and $\frac{(z(L_c I^\sigma f(z))')'}{(L_c I^\sigma f(z))'} \neq \lambda, z \in U$ then $f \in CV_\sigma(\eta, \lambda)$ implies that $I^\sigma f \in CV_c(\eta, \lambda)$.

We claim the similar results may be hold for meromorphic p-valent functions with alternating coefficient. For more information see [2].

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